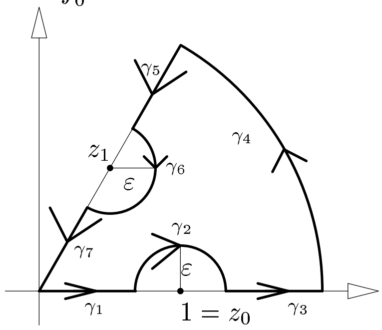
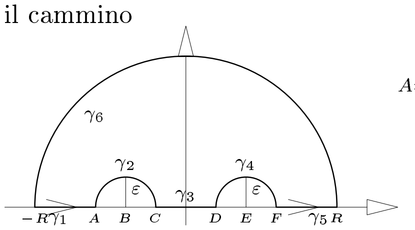
• h) $\displaystyle\int_0^{+\infty}\frac{dx}{x^{2n}-1}$



$$\begin{cases} \gamma_1(t)=(t,0),\ 0\le t\le 1-\varepsilon \qquad \gamma_2(t)=z_0+\varepsilon e^{-i\vartheta}\ \ -\pi\le\vartheta\le 0 \\ \gamma_3(t)=(t,0),\ 1+\varepsilon\le t\le R \qquad \gamma_4(t)=Re^{i\vartheta}\ 0\le\vartheta\le\dfrac{\pi}{n} \\ \gamma_5(t)=-te^{i\frac{\pi}{n}},\ -R\le t\le -(1+\varepsilon) \quad \gamma_7(t)=-te^{i\frac{\pi}{n}},\ -(1-\varepsilon)\le t\le 0 \\ \gamma_6(t)=z_1+\varepsilon e^{-i\vartheta},\ -\dfrac{n-1}{n}\pi\le\vartheta\le\dfrac{\pi}{n}, \qquad z_1=e^{i\frac{\pi}{n}} \end{cases}$$

Dalla figura si ha $\displaystyle\lim_{R\to+\infty}\lim_{\varepsilon\to0}\int_\Gamma f(z)dz = \int_0^{+\infty}\frac{dt}{t^{2n}-1} - i\pi\lim_{t\to1}\frac{t-1}{t^{2n}-1} + \int_{-\infty}^0\frac{-dte^{-\frac{\pi}{n}}}{t^{2n}-1} - i\pi\lim_{z\to z_1}\frac{z-z_1}{z^{2n}-1} = 2\pi i\sum_k Resf(z_k)$ e la somma dà zero in quanto non vi sono singolarità interne al cammino. Dunque si ottiene $I(1-e^{i\frac{\pi}{n}}) = i\pi(\frac{1}{2n}+\frac{1}{2n}z_1) = \frac{i\pi}{2n}(1+e^{i\frac{\pi}{n}})$ e quindi $I=-\frac{\pi}{n}\cot\frac{\pi}{2n}$

Si poteva anche scrivere $\displaystyle I=\frac12\int_{-\infty}^{+\infty}\frac{dx}{x^{2n-1}} = \frac12\int_{-\infty}^{+\infty}\frac{dx}{(x-1)(x+1)P_n(x)}$ con $P(x)=\dfrac{x^{2n-1}}{(x-1)(x+1)}$ e utilizzare il cammino



$A{=}-i-\varepsilon,\ B{=}-i\ C{=}-i+\varepsilon\ D{=}i-\varepsilon\ E{=}i\ F{=}i+\varepsilon$

$z_k = e^{i\frac{k}{n}\pi}$ $k=0,\dots,2n-1$ sono le radici del polinomio $z^{2n-1}-1=0$ e solamente $k=1,\dots,n-1$ appartengono al semipiano superiore mentre $z_0=1$, $z_n=-1$. Abbiamo (con n dispari) $\displaystyle I=\pi i\sum_{k=1}^{n-1}Resf(z_k) + i\pi\lim_{z\to z_0}\frac{1}{\Pi_{k=1}^{2n-1}(z-z_k)} + i\pi\lim_{z\to z_n}\frac{1}{\Pi_{k=0,k\ne n}^{2n-1}(z-z_k)}$. $\displaystyle\Pi_{k=1}^{2n-1}(z-z_k)=\frac{z^{2n}-1}{z-1}=\lim_{z\to1}2nz^{2n-1}=2n.$

$\displaystyle\Pi_{\substack{k=0\\k\ne n}}^{2n-1}(z-z_k)=\frac{z^{2n}-1}{z-z_n}=\lim_{z\to-1}2nz^{2n-1}=-2n$. Ne segue che $\displaystyle I=\pi i\sum_{k=1}^{n-1}Resf(z_k)=\pi i\sum_{k=1}^{n-1}\lim_{z\to z_k}\frac{z-z_k}{z^{2n}-1}=$

$$\pi i\sum_{k=1}^{n-1}\lim_{z\to z_k}\frac{1}{2nz^{2n-1}} = \pi i\sum_{k=1}^{n-1}z_k\frac{1}{2n} = \frac{\pi i}{2n}2i\sum_{k=1}^{\frac{n-1}{2}}\sin\frac{k}{n}\pi = -\frac{\pi}{2ni}\Big(\frac{e^{i\frac{\pi}{n}}-e^{i(\frac{\pi}{n}+\frac{n+1}{2})}}{1-e^{i\frac{\pi}{n}}} - \frac{e^{-i\frac{\pi}{n}}-e^{-i(\frac{\pi}{n}+\frac{n+1}{2})}}{1-e^{-i\frac{\pi}{n}}}\Big)=$$

$$=-\frac{\pi}{2ni}\Big(\frac{e^{i\frac{\pi}{n}}-i}{-2i\sin\frac{\pi}{2n}} - \frac{e^{-i\frac{\pi}{n}}-(-i)}{2i\sin\frac{\pi}{2n}}\Big) = -\frac{\pi}{2in}(-)\frac{2\cos\frac{\pi}{2n}}{2i\sin\frac{\pi}{2n}} = -\frac{\pi}{2n}\cot\frac{\pi}{2n}$$

Sia ora n pari. $\displaystyle I=\frac{\pi}{2n}(i+\sum_{k=1}^{\frac{n}{2}-1}2i\sin\frac{k}{n}\pi) = \frac{\pi}{2n}(i+\frac{e^{i\frac{\pi}{n}}-e^{i\frac{\pi}{n}\frac{n}{2}}}{1-e^{i\frac{\pi}{n}}} - \frac{e^{-i\frac{\pi}{n}}-e^{-i\frac{\pi}{n}\frac{n}{2}}}{1-e^{-i\frac{\pi}{n}}}) = \frac{\pi i}{2n}(i+\frac{e^{i\frac{\pi}{n}}-i}{1-e^{i\frac{\pi}{n}}} - \frac{e^{-i\frac{\pi}{n}}+i}{1-e^{-i\frac{\pi}{n}}})=$

$$\frac{\pi i}{2n}(i+\frac{e^{i\frac{\pi}{2n}}-ie^{i\frac{\pi}{2n}}}{-2i\sin\frac{\pi}{2n}} - \frac{e^{-i\frac{\pi}{2n}}+ie^{i\frac{\pi}{2n}}}{2i\sin\frac{\pi}{2n};}) = \frac{\pi i}{2n}\frac{1}{\sin\frac{\pi}{2n}}(-\frac{1}{i}\cos\frac{\pi}{2n}+i\sin\frac{\pi}{2n}-i\sin\frac{\pi}{2n}) = -\frac{\pi}{2n}\cot\frac{\pi}{2n}$$